

A RIGIDITY RESULT FOR SOME PARABOLIC GERMS

LUNA LOMONACO AND SABYASACHI MUKHERJEE

ABSTRACT. The goal of this note is to prove a rigidity result for unicritical polynomials with parabolic cycles.

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Parabolic-like maps	3
4. Proof of the Theorem	5
References	8

1. INTRODUCTION

Let $f_c(z) = z^d + c$. We will denote the Julia and filled-in Julia sets of f_c by $J(f_c)$ and $K(f_c)$ respectively. The degree d multibrot set \mathcal{M}_d is defined as the set of all complex parameters c for which $K(f_c)$ is connected.

A parameter c of \mathcal{M}_d is called a parabolic parameter if f_c has a periodic cycle with multiplier a root of unity. We will prove a rigidity principle concerning parabolic germs obtained as restrictions of suitable iterates of f_c where c is a parabolic parameter of \mathcal{M}_d . Every parabolic parameter of \mathcal{M}_d is the root point of a hyperbolic component. For $i = 1, 2$, let c_i be the root point of a hyperbolic component H_i of period n_i of \mathcal{M}_d , z_i be the characteristic parabolic point (i.e. the parabolic periodic point on the boundary of the critical value Fatou component) of f_{c_i} . It is worthwhile to note that under the above assumptions, $(f_{c_i}^{on_i})'(z_i) = 1$; i.e. the restriction of $f_{c_i}^{on_i}$ in a neighborhood of z_i is a parabolic germ with multiplier 1. We show that:

Theorem 1.1 (Parabolic Germs Determine Parabolic Parameters). *If there exist small neighborhoods N_1 and N_2 of z_1 and z_2 (in the dynamical planes of c_1 and c_2 respectively) such that $f_{c_1}^{on_1}|_{N_1}$ and $f_{c_2}^{on_2}|_{N_2}$ are conformally conjugate, then f_{c_1} and f_{c_2} are affinely conjugate.*

By the classical theory of conformal conjugacy classes of parabolic germs [Eca75, Vor81], there is an infinite-dimensional family of conformally different parabolic germs. On the other hand, there is only a one-parameter family of unicritical complex polynomials of a given degree. Hence it seems extremely unlikely that two different unicritical polynomials (of the same degree) would have conformally conjugate parabolic germ restrictions (compare [CEP15, §3]). The above theorem confirms this suspicion by showing that if suitable iterates of two unicritical parabolic polynomials have conformally conjugate ‘tangent to identity’ parabolic germ restrictions, then the two polynomials are indeed affinely conjugate.

The paper is organized as follows. We will review some theory about parabolic germs in Section 2, and we will review the theory of parabolic-like maps (a parabolic analogue of polynomial-like maps introduced by the first author in [Lom15]) in Section 3. We will prove Theorem 1.1 in Section 4 by combining a rigidity result about parabolic-like maps (see Lemma 4.1) and a local to global principle that concludes a global statement about two polynomials from a local conformal conjugacy information between them (see Lemma 4.2).

2. PRELIMINARIES

Recall that for a parabolic germ, the attracting and repelling Fatou coordinates that conjugate the dynamics of the germ to translation by $+1$, are defined in the various attracting and repelling petals respectively. For a general parabolic germ, these various Fatou coordinates do not agree on their common domains of definition. Loosely speaking, horn maps are objects that record the difference between a pair of (adjacent) attracting and repelling Fatou coordinates. Horn maps completely determine the conformal conjugacy classes of parabolic germs, and conversely, if two parabolic germs are conformally conjugate, then they have the same horn maps [Eca75, Vor81]. In our setting, where the parabolic germs are restrictions of globally defined polynomials, the horn maps have a natural maximal domain of definition (the extension is obtained by iterating the dynamics). One of the crucial properties of these extended horn maps that we will have need for is that they are finite-type maps whose critical values are completely determined by the conformal positions of the critical points of the polynomial [BE02, Proposition 4][Eps93].

We say that a parabolic parameter c is primitive (or simple) if each parabolic periodic point of f_c has a single attracting petal. Equivalently, a primitive parabolic parameter c lies on the boundary of a unique hyperbolic component of \mathcal{M}_d , and is the root thereof. In this case, the period of the corresponding hyperbolic component (on whose boundary c lies) is equal to the period of the parabolic cycle of f_c .

A parabolic parameter c of \mathcal{M}_d is called satellite if each parabolic periodic point of f_c has at least two attracting petals. Equivalently, a satellite parabolic parameter c is the (unique) common boundary point of two hyperbolic components H and H' of \mathcal{M}_d , one of which, say H' , has c as its root. In this case, the period of H' is strictly greater than the period of the parabolic cycle of f_c (compare [EMS16]).

Since the proof of Theorem 1.1 in the case when both c_1 and c_2 are primitive parameters is essentially present in [IM16, Theorem 1.4], we will be concerned here with the satellite case. Let the period of the parabolic cycle of f_{c_i} be k_i (so c_i sits on the boundary of a hyperbolic component of period k_i and a hyperbolic component of period n_i). Set $q_i := n_i/k_i$. Recall that $f_{c_i}^{n_i}(z) = z + a_i(z - z_i)^{q_i+1} + O((z - z_i)^{q_i+2})$ for some $a_i \in \mathbb{C}$, as the Taylor expansion of $f_{c_i}^{n_i}$ near z_i . Furthermore, q_i is the number of attracting petals at the parabolic point z_i , and these petals are permuted by $f_{c_i}^{k_i}$. If the parabolic germs of $f_{c_i}^{n_i}$ (for $i = 1, 2$) are conformally conjugate, then we have $q_1 = q_2$ (in fact, q_i is a topological conjugacy invariant of the parabolic germ). Therefore, the two polynomials f_{c_1} and f_{c_2} have the same number of petals at each parabolic periodic point.

3. PARABOLIC-LIKE MAPS

The theory of parabolic-like maps extends the theory of polynomial-like maps to objects with a parabolic external class. For any polynomial map P on the Riemann sphere $\widehat{\mathbb{C}}$, infinity is a superattracting fixed point, and the filled Julia set K_P is the complement of the basin of attraction of infinity $\mathcal{A}(\infty)$, that is $K_P = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$. So the preimage of a topological disk U (nice enough, for example bounded by an equipotential) containing K_P is a topological disk U' compactly contained in U , and $P|_{U'} : U' \rightarrow U$ is a proper holomorphic map of degree $d = \deg(P)$. The triple (P, U', U) is a (trivial) example of a polynomial-like map. Formally, a $(\deg d)$ polynomial-like map is a triple (f, U', U) where U' and U are topological disks, $U' \subset\subset U$ and $f : U' \rightarrow U$ is a $(\deg d)$ proper holomorphic map [DH85]. The filled Julia set K_f of a polynomial-like map is the set of points which never leave U' under iteration (for a polynomial P , this is just K_P). With any degree d polynomial-like map, one can associate a degree d covering of the unit circle $h_f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ which encodes the dynamics of the polynomial-like map outside its filled Julia set. The map h_f is called the *external map* of the polynomial-like map f . The external map of a polynomial-like map is strictly expanding, with all periodic points repelling, and it is defined up to real-analytic diffeomorphisms of the circle. In this way a polynomial-like map can be considered as a union of two different dynamical systems: the filled Julia set K_P and the external map h_f . By replacing the external map of a degree d polynomial-like map with the map $z \rightarrow z^d$ (which is an external map for a degree d polynomial), Douady and Hubbard proved that every degree d polynomial-like map is hybrid equivalent to a polynomial of the

same degree (where a hybrid equivalence is a quasiconformal conjugacy φ with $\bar{\partial}\varphi = 0$ on K_f), and that this polynomial is unique if K_f is connected.

A parabolic-like map is an object similar to a polynomial-like map, in the sense that it can be considered as the union of two different dynamical systems: the filled Julia set and the external map [Lom15]. However, the external map of a parabolic-like map contains a parabolic fixed point, which complicates the setting considerably.

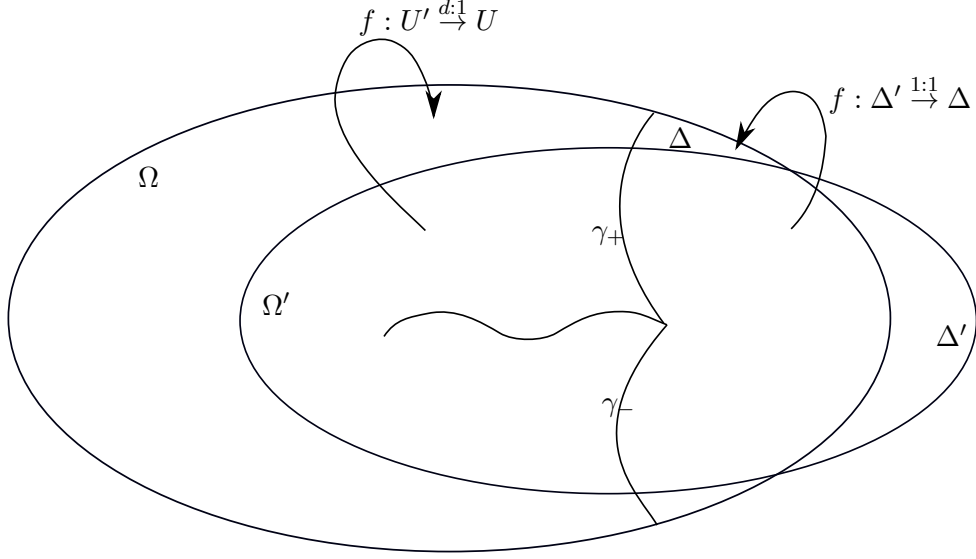


FIGURE 1. For a parabolic-like map (f, U', U, γ) the arc γ divides U' and U into Ω', Δ' and Ω, Δ respectively. These sets are such that Ω' is compactly contained in U , $\Omega' \subset \Omega$, $f : \Delta' \rightarrow \Delta$ is an isomorphism and Δ' contains at least one attracting fixed petal of the parabolic fixed point.

Definition. (Parabolic-like maps) A *parabolic-like map* of degree $d \geq 2$ is a 4-tuple (f, U', U, γ) where

- U' and U are open subsets of \mathbb{C} , with U' , U and $U \cup U'$ isomorphic to a disc, and U' not contained in U ,
- $f : U' \rightarrow U$ is a proper holomorphic map of degree $d \geq 2$ with a parabolic fixed point at $z = z_0$ of multiplier 1,
- $\gamma : [-1, 1] \rightarrow \bar{U}$ is an arc with $\gamma(0) = z_0$, forward invariant under f , C^1 on $[-1, 0]$ and on $[0, 1]$, and such that

$$f(\gamma(t)) = \gamma(dt), \quad \forall -\frac{1}{d} \leq t \leq \frac{1}{d},$$

$$\gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U', \quad \gamma(\pm 1) \in \partial U.$$

It resides in repelling petal(s) of z_0 and it divides U' and U into Ω', Δ' and Ω, Δ respectively, such that $\Omega' \subset\subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \rightarrow \Delta$ is an isomorphism (see Figure 1) and Δ' contains at least one attracting fixed petal of z_0 . We call the arc γ a *dividing arc*.

The filled Julia set K_f of a parabolic-like map (f, U', U, γ) is the set of points which never leave $\Omega' \cup \{z_0\}$ under iteration. The model family in degree 2 is given by the family of quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity (normalized by having critical points at ± 1), this is $Per_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A, A \in \mathbb{C}\}$. The filled Julia set for P_A with $A \neq 0$ is the complement of the parabolic basin of infinity $\mathcal{A}_A(\infty)$, so (as for polynomial-like mappings) $K_A = \widehat{\mathbb{C}} \setminus \mathcal{A}_A(\infty)$ (for $A = 0$ we need to make a choice, since both the left and right half planes are parabolic basins. We set $K_{P_0} = \overline{\mathbb{H}}_l$). The map $h_2(z) = \frac{z^2+1/3}{z^2/3+1}$ is an external map for every P_A , $A \in \mathbb{C}$ [Lom15, Proposition 4.2]. By replacing the external map of a degree 2 parabolic-like map with the map h_2 , the first author proved [Lom15] that every degree 2 parabolic-like map is hybrid equivalent to a member of the family $Per_1(1)$, and that this member is unique if K_f is connected. For more detailed studies on parabolic-like maps, consult [Lom15] for a dynamical description, [Lom14a] for a parameter space (of degree 2 analytic families of parabolic-like maps) description, and [Lom14b] for an easy discussion on the results contained in the previous two articles.

4. PROOF OF THE THEOREM

The root of every satellite hyperbolic component of the multibrot set \mathcal{M}_d admits a parabolic-like restriction (see [Lom15, §3.1, Example 3] for details of the construction in the case $d = 2$, the case $d > 2$ being similar).

Lemma 4.1 (Rigidity of Parabolic-like Mappings). *Let c_1 and c_2 be the root points of two satellite hyperbolic components H_1 and H_2 (of period n_1 and n_2 respectively) of the Multibrot set \mathcal{M}_d . If the parabolic-like mappings defined by the restrictions of $f_{c_1}^{on_1}$ and $f_{c_2}^{on_2}$ (around their characteristic Fatou components) are conformally conjugate, then $c_1 = c_2$ up to affine conjugacy.*

Proof. Let c_i be the root of a satellite component of period n_i (attached at c_1 to an hyperbolic component of period k_i). Then $q_i = n_i/k_i$ is the number of attracting petals at the parabolic point z_i of f_{c_i} . Let us start by noticing that if the parabolic-like mappings defined by the restrictions of $f_{c_1}^{on_1}$ and $f_{c_2}^{on_2}$ are conformally conjugate, then the parabolic germs of $f_{c_i}^{on_i}$ (for $i = 1, 2$) are conformally conjugate, and hence $q_1 = q_2 = q$.

We label the Fatou components of f_{c_i} touching at the characteristic parabolic point z_i counter-clockwise such that U_i^1 is the Fatou component of f_{c_i} containing the critical value c_i . Since c_i is the root of a satellite component attached to another hyperbolic component of period k_i , the polynomial $f_{c_i}^{ok_i}$ has a polynomial-like restriction (h_i, V_i', V_i) that is hybrid equivalent to

some (degree d) $\frac{p_i}{q}$ -rabbit (basilica if $q = 2$) parameter on the boundary of the principal hyperbolic component of \mathcal{M}_d (more precisely, $f_{c_i}^{\circ k_i}$ has a polynomial-like restriction h_i that is hybrid equivalent to some polynomial $f_{c'_i}$ with a fixed point of multiplier $e^{\frac{2\pi i p_i}{q}}$).

Let η be a conformal conjugacy between the parabolic-like restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ in neighborhoods of $\overline{U_1^1}$ and $\overline{U_2^1}$ (respectively). A priori, η is defined only in a neighborhood W of $\overline{U_1^1}$. We will show that using the dynamics, η can be extended as a conformal conjugacy from a neighborhood of $K(h_1)$ to a neighborhood of $K(h_2)$.

Choose an neighborhood W_0 of z_1 with $W_0 \subset W$ and such that W_0 does not contain any critical point of $f_{c_1}^{\circ n_1}$. Since W_0 intersects the Julia set of

f_{c_1} , we have that $\bigcup_{s=1}^{\infty} f_{c_1}^{\circ sn_1}(W_0) \supset K(h_1)$. Now fix $s \in \mathbb{N}$. Since the domain of η contains W_0 , we can use the functional equation $\eta \circ f_{c_1}^{\circ sn_1} = f_{c_2}^{\circ sn_2} \circ \eta$ to push forward η to a holomorphic conjugacy $\eta_s : f_{c_1}^{\circ sn_1}(W_0) \rightarrow f_{c_2}^{\circ sn_2}(\eta(W_0))$ between $f_{c_1}^{\circ n_1}$ and $f_{c_1}^{\circ n_1}$ (possibly after shrinking W_0 to ensure that $f_{c_1}^{\circ sn_1}$ has no critical point in W_0). Since $W_0 \cap U_1^1 \neq \emptyset$, $f_{c_1}^{\circ sn_1}(U_1^1) = U_1^1$, and η is already a conjugacy on U_1^1 , it follows from the construction that $\eta_s \equiv \eta$ on the non-empty open set $f_{c_1}^{\circ sn_1}(W_0) \cap U_1^1$. This proves that η_s extends the map η . Since we can perform this operation for each $s \in \mathbb{N}$, uniqueness of analytic continuation yields a holomorphic map η defined on a neighborhood of $K(h_1)$ such that it conjugates $f_{c_1}^{\circ n_1}$ to $f_{c_2}^{\circ n_2}$.

Therefore, η is a conformal conjugacy between the polynomial-like maps $h_1^{\circ q}$ and $h_2^{\circ q}$. By [IM16, Corollary 10.2], we have $c_1 = c_2$ up to affine conjugacy. \square

We now proceed to promote the germ conjugacy between $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ to a conformal conjugacy between two suitable parabolic-like mappings. We label the Fatou components of f_{c_i} touching the parabolic point z_i counter-clockwise such that U_i^1 is the Fatou component of f_{c_i} containing the critical value c_i . In order to investigate the consequences of such a conformal conjugacy between two parabolic germ restrictions, we will need to use the concept of extended horn maps, which are the natural maximal extensions of horn maps. A comprehensive account of horn maps, and their mapping properties can be found in [BE02, Shi00], so we do not include the definition here.

Lemma 4.2. *Let c_1 and c_2 be the root points of two satellite hyperbolic components H_1 and H_2 (of period n_1 and n_2 respectively) of the Multibrot set \mathcal{M}_d , and z_1 and z_2 be the characteristic parabolic points of f_{c_1} and f_{c_2} (respectively). Then the following are equivalent.*

- *The parabolic-like mappings defined by the restrictions of $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ (around U_1^1 and U_2^1 respectively) are conformally conjugate.*

- *The (tangent-to-identity) parabolic germs given by the restrictions of $f_{c_1}^{on_1}$ and $f_{c_2}^{on_2}$ (around z_1 and z_2 respectively) are conformally conjugate.*

Proof. Conformal conjugacy of the parabolic-like maps clearly implies conformal conjugacy of the corresponding germs. So we only need to show that when $g_1 := f_{c_1}^{on_1}|_{N_1}$ and $g_2 := f_{c_2}^{on_2}|_{N_2}$ are conformally conjugate by some local biholomorphism $\varphi_1 : N_1 \rightarrow N_2$ (where N_i is a small neighborhood of z_i), the parabolic-like maps $f_{c_1}^{on_1}$ and $f_{c_2}^{on_2}$ (around U_1^1 and U_2^1 respectively) are also conformally conjugate. We now proceed to prove this.

Note that φ_1 must map an attracting petal $\mathcal{P}_{c_1}^{\text{att},1} \subset N_1 \cap U_1^1$ to some attracting petal $\mathcal{P}_{c_2}^{\text{att},k} \subset N_2 \cap U_2^k$, and so $\varphi := f_{c_2}^{ok_2(1-k)} \circ \varphi_1$ is a conformal conjugacy between g_1 and g_2 such that it maps $\mathcal{P}_{c_1}^{\text{att},1}$ to a petal $\mathcal{P}_{c_2}^{\text{att},1}$ (this is no more than a matter of convenience).

For $k \in \mathbb{Z}/q\mathbb{Z}$, if $\psi_{c_2}^{\text{att},k}$ is an extended attracting Fatou coordinate for $f_{c_2}^{on_2}$ in U_2^k , then there exist extended attracting Fatou coordinate $\psi_{c_1}^{\text{att},k}$ for $f_{c_1}^{on_1}$ in U_1^k such that $\psi_{c_1}^{\text{att},k} = \psi_{c_2}^{\text{att},k} \circ \varphi$ in their common domain of definitions. Similarly, we can transport the extended repelling Fatou coordinates of $f_{c_2}^{on_2}$ at z_2 by φ to define extended repelling Fatou coordinates of $f_{c_1}^{on_1}$ at z_1 . It follows that with these choices of Fatou coordinates the extended horn maps $h_{c_i,k}^\pm$ of $f_{c_i}^{n_i}$ at z_i coincide, this is $h_{c_1,k}^\pm = h_{c_2,k}^\pm$. By [BE02, Proposition 4], for $i = 1, 2$, $h_{c_i,1}^+$ is a ramified covering with the unique critical value $\Pi(\psi_{c_i}^{\text{att},1}(c_i))$. Hence $\Pi(\psi_{c_1}^{\text{att},1}(c_1)) = \Pi(\psi_{c_2}^{\text{att},1}(c_2))$. Therefore, $\psi_{c_1}^{\text{att},1}(c_1) - \psi_{c_2}^{\text{att},1}(c_2) = r \in \mathbb{Z}$ (more precisely, r is a multiple of q). Normalize these attracting Fatou coordinates such that $\psi_{c_1}^{\text{att},1}(c_1) = 0$ and $\psi_{c_2}^{\text{att},1}(c_2) = -r$.

Define on $\mathcal{P}_{c_2}^{\text{att},1}$ the map $\tilde{\psi}_{c_2}^{\text{att},1} = \psi_{c_2}^{\text{att},1} \circ g_2^{\circ(r)}$, so $\tilde{\psi}_{c_2}^{\text{att},1}$ is an attracting Fatou coordinate for $f_{c_2}^{on_2}$ at z_2 such that, for all $x \in \mathcal{P}_{c_2}^{\text{att},1}$, $\tilde{\psi}_{c_2}^{\text{att},1}(x) - \psi_{c_2}^{\text{att},1}(x) = r$. By analytic continuation, we have an extended attracting Fatou coordinate $\tilde{\psi}_{c_2}^{\text{att},1} : U_2^1 \rightarrow \mathbb{C}$ for $f_{c_2}^{on_2}$ such that

$$\tilde{\psi}_{c_2}^{\text{att},1}(c_2) = \psi_{c_2}^{\text{att},1}(c_2) \circ (f_{c_2}^{on_2})^{\circ(r)}(c_2) = \psi_{c_2}^{\text{att},1}(c_2) + r = 0 = \psi_{c_1}^{\text{att},1}(c_1).$$

Define $\eta := (\tilde{\psi}_{c_2}^{\text{att},1})^{-1} \circ \psi_{c_1}^{\text{att},1} : N_1 \cap U_1^1 \rightarrow N_2 \cap U_2^1$, then η is a conformal conjugacy between $f_{c_1}^{on_1}|_{N_1}$ and $f_{c_2}^{on_2}|_{N_2}$ which extends by iterated lifting to a conformal conjugacy $\eta : U_1^1 \rightarrow U_2^1$ between $f_{c_1}^{on_1}|_{U_1^1}$ and $f_{c_2}^{on_2}|_{U_2^1}$.¹ Abusing notation, we will denote this extended conjugacy by η . Since the basin boundaries are locally connected, by Caratheodory's theorem the conformal

¹Here is an alternative route to extend η to the entire Fatou component. We can choose Riemann maps $\varphi_{c_i} : U_i^1 \rightarrow \mathbb{D}$ with $\varphi_{c_i}(c_i) = 0$ such that φ_{c_i} conjugates $f_{c_i}^{on_i}|_{U_i^1}$ to the Blaschke product $B(z) = \frac{3z^2+1}{3+z^2}$. An easy computation in Fatou coordinates now shows that $\varphi_{c_2}^{-1} \circ \varphi_{c_1}$ extends the local conjugacy η to the entire immediate basin U_1^1 such that it conjugates $f_{c_1}^{on_1}$ on U_1^1 to $f_{c_2}^{on_2}$ on U_2^1 .

conjugacy η extends as a homeomorphism from ∂U_1^1 onto ∂U_2^1 . Note also that by definition, $\eta = g_2^{\circ(-r)} \circ \varphi$ in their common domain of definition. Therefore, η is defined in a neighborhood V of the point z_1 , and continues to be a conjugacy between the germs g_1 and g_2 .

We can now extend this conformal conjugacy to a conformal conjugacy η between a neighborhood of \overline{U}_1^1 and a neighborhood of \overline{U}_2^1 following the proof of Lemma 4.3 in [IM16]. We include the details for the reader. By Montel's theorem, $\bigcup_{s \in \mathbb{N}} f_{c_1}^{\circ sn_1}(V \cap \partial U_1^1) = \partial U_1^1$, and since none of the $f_{c_1}^{\circ sn_1}$ has a critical point on ∂U_1^1 , we can extend η in a neighborhood of each point of ∂U_1^1 by using the functional equation $\eta \circ f_{c_1}^{\circ sn_1} = f_{c_2}^{\circ sn_2} \circ \eta$. Uniqueness of analytic continuation yields an analytic extension of η in a neighborhood of \overline{U}_1^1 . Moreover, the extension is a proper degree 1 holomorphic map, and is a conformal conjugacy between $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$. This shows that the parabolic-like mappings defined by $f_{c_1}^{\circ n_1}$ and $f_{c_2}^{\circ n_2}$ in neighborhoods of \overline{U}_1^1 and \overline{U}_2^1 (respectively) are conformally conjugate. \square

Proof of Theorem 1.1. Since the multiplicity of a parabolic germ is a topological invariant, it follows that either both c_1 and c_2 are primitive, or both of them are satellite. In the former case, the period of H_i is equal to the period of the parabolic cycle of f_{c_i} . Therefore, the conclusion follows from [IM16, Theorem 1.4]. On the other hand, when both the c_i are satellite parabolic parameters, the result follows from Lemma 4.1 and Lemma 4.2. \square

REFERENCES

- [BE02] Xavier Buff and Adam L. Epstein. A parabolic Pommerenke-Levin-Yoccoz inequality. *Fund. Math.*, 172:249–289, 2002.
- [CEP15] Arnaud Chéritat, Adam Lawrence Epstein, and Carsten Lunde Petersen. Perspectives on parabolic points in holomorphic dynamics. <http://www.birs.ca/workshops/2015/15w5082/report15w5082.pdf>, 2015. Conference report.
- [DH85] Adrien Douady and John H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. Ec. Norm. Sup.*, 18:287–343, 1985.
- [Eca75] Jean Ecalle. Théorie itérative : introduction à la théorie des invariants holomorphes. *J. Math. Pures Appl. (9)*, 54:183–25, 1975.
- [EMS16] D. Eberlein, S. Mukherjee, and D. Schleicher. Rational parameter rays of the multibrot sets. In *Dynamical Systems, Number Theory and Applications*, chapter 3, pages 49–84. World Scientific, 2016. http://dx.doi.org/10.1142/9789814699877_0003.
- [Eps93] Adam Epstein. *Towers of Finite Type Complex Analytic Maps*. PhD thesis, The City University of New York, 1993.
- [IM16] Hiroyuki Inou and Sabyasachi Mukherjee. Discontinuity of straightening in antiholomorphic dynamics. <https://arxiv.org/abs/1605.08061>, 2016.
- [Lom14a] Luna Lomonaco. Parameter space for families of parabolic-like mappings. *Advances in Mathematics*, 261C:200–219, 2014.
- [Lom14b] Luna Lomonaco. Results about parabolic-like mappings. *Analysis in Theory and Applications*, 30:120–129, 2014.

- [Lom15] Luna Lomonaco. Parabolic-like mappings. *Ergodic Theory and Dynamical Systems*, 35:2171–2197, 2015.
- [Shi00] Mitsuhiro Shishikura. Bifurcation of parabolic fixed points. In *The Mandelbrot Set, Theme and Variations*, London Mathematical Society Lecture Note Series (No. 274), pages 325–364. Cambridge University Press, 2000.
- [Vor81] S. M. Voronin. Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. *Funktsional. Anal. i Prilozhen*, 15:1–17, 1981.

E-mail address: lunalomonaco@gmail.com

INSTITUTE FOR MATHEMATICAL SCIENCES, STONY BROOK UNIVERSITY, NY, 11794,
USA

E-mail address: sabya@math.stonybrook.edu